

MODIFIED QUATERNION AND OCTONION
DIVISION ALGEBRAS

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Abstract

We consider two classes of real division algebras that arise from modifications of the quaternions and the octonions. Since a *division algebra* is defined to be an algebra in which the equations $ux = v$ and $yu = v$ have unique solutions for all vectors v and nonzero vectors u , it follows that an algebra is a division algebra if and only if left translation by any nonzero vector u is invertible. That can be verified by checking that its determinant is positive definite. Our approach is elementary and derives from routine considerations of the algebras' multiplication tables. The quaternions and octonions are obtained from my tables by setting all $a_i = 1$ and all $b_{ij} = 0$.

Modified Quaternion Algebras

Let $\{1, e_1, e_2, e_3\}$ be a basis for the modified quaternions defined by the multiplication Table (1).

I have suppressed the identity row and column and the basis element, 1. For example, $b_{12} + e_3$ is shorthand for $b_{12}1 + e_3$ and $a + be_1 + ce_2 + de_3$ is shorthand for $a1 + be_1 + ce_2 + de_3$.

	1	e_1	e_2	e_3
(1)	e_1	$-a_1$	$b_{12} + e_3$	$b_{13} - e_2$
	e_2	$-b_{12} - e_3$	$-a_2$	$b_{23} + e_1$
	e_3	$-b_{13} + e_2$	$-b_{23} - e_1$	$-a_3$

Let L denote left translation by the vector $u = a + be_1 + ce_2 + de_3$.

Then $\det(L) = (a^2 + b^2 + c^2 + d^2)[a^2 + a_1b^2 + a_2c^2 + a_3d^2]$ independent of b_{12}, b_{13}, b_{23} . We have

Theorem 1. The modified quaternion algebra in Table (1) is a division algebra if and only if $a_1, a_2, a_3 > 0$.

Definition. An algebra A is *alternative* if for all x, y in A , $x(yy) = (xy)y$ and $(xx)y = x(xy)$.

Theorem 2. The algebra with Table (1) is alternative if and only if it is the quaternions.

Proof. We have $e_i(e_j e_j) = -a_j e_i$ and for some k , $(e_i e_j) e_j = (b_{ij} \pm e_k) e_j = b_{ij} \pm b_{ij} e_j - e_i$. It follows that the algebra with Table (1) is alternative if and only if $b_{12} = b_{13} = b_{23} = 0$ and $a_1 = a_2 = a_3 = 1$.

Modified Octonion Algebras

In the introduction of his “octavic system” in 1881 [1], Cayley considers the 8-dimensional real vector space $\mathbb{R}^8$ with basis vectors symbolized in a confusing way no one would choose today. His basis consists of vectors denoted by the symbols $0, 1, 2, 3, 4, 5, 6, 7$ where 0 denotes what he refers to as the “positive unity,” *i.e.*, what we denote by 1. He refers to the vectors $1, 2, 3, 4, 5, 6, 7$ as “the seven imaginaries.”

This vector space becomes an algebra under a multiplication Cayley adapts from the quaternions. To begin, $0^2 = 0$ and $1^2 = 2^2 = 3^2 = 4^2 = 5^2 = 6^2 = 7^2 = -0$. Recall that Cayley uses the symbol 0 to represent 1 and the symbols $1, \dots, 7$ to represent basis vectors. He then sets

$$\varepsilon_i = \pm 1 \text{ for } i = 1, \dots, 7.$$

To define the product of distinct pairs of vectors, he uses a compressed notation that in the quaternions would be written “ $ijk = \varepsilon$.” That compact equation summarizes the six equations

$jk = \varepsilon i, ki = \varepsilon j, ij = \varepsilon k, kj = -\varepsilon i, ik = -\varepsilon j, \text{ and } ji = -\varepsilon k$, where, for the quaternions $\varepsilon = +1$. This representation corresponds to the familiar circle mnemonic for quaternion multiplication:



The octonion multiplication table is completed by the seven compact equations:

$$123 = \varepsilon_1, 145 = \varepsilon_2, 167 = \varepsilon_3, 246 = \varepsilon_4, 257 = \varepsilon_5, 347 = \varepsilon_6, 356 = \varepsilon_7.$$

We next present multiplication in tabular form:

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	-0	$\varepsilon_1 3$	$-\varepsilon_1 2$	$\varepsilon_2 5$	$-\varepsilon_2 4$	$\varepsilon_3 7$	$-\varepsilon_3 6$
2	2	$-\varepsilon_1 3$	-0	$\varepsilon_1 1$	$\varepsilon_4 6$	$\varepsilon_5 7$	$-\varepsilon_4 4$	$-\varepsilon_5 5$
3	3	$\varepsilon_1 2$	$-\varepsilon_1 1$	-0	$\varepsilon_6 7$	$\varepsilon_7 6$	$-\varepsilon_7 5$	$-\varepsilon_6 4$
4	4	$-\varepsilon_2 5$	$-\varepsilon_4 6$	$-\varepsilon_6 7$	-0	$\varepsilon_2 1$	$\varepsilon_2 2$	$\varepsilon_6 3$
5	5	$\varepsilon_2 4$	$-\varepsilon_5 7$	$-\varepsilon_7 6$	$-\varepsilon_2 1$	-0	$\varepsilon_7 3$	$\varepsilon_5 2$
6	6	$-\varepsilon_3 7$	$\varepsilon_4 4$	$\varepsilon_7 5$	$-\varepsilon_4 2$	$-\varepsilon_7 3$	-0	$\varepsilon_3 1$
7	7	$\varepsilon_3 6$	$\varepsilon_5 5$	$\varepsilon_6 4$	$-\varepsilon_6 3$	$-\varepsilon_5 2$	$-\varepsilon_3 1$	-0

Cayley's interest was in solving the "eight square" problem [2]. To do this he determined conditions on the ε 's so that a sum of eight squares will be the product of two sums of eight squares. He first determines that without loss of generality one can take $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ and then it follows that $-\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \pm 1$.

Either of these two choices of the ε_i guarantee that the algebra is a division algebra. See [2]. For specificity, I select $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$ and $\varepsilon_5 = \varepsilon_6 = \varepsilon_7 = -1$ and then present a modification of Cayley's multiplication table as follows, suppressing the identity row and column and the basis vector, 1.

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_1	$-a_1$	$b_{12} + e_3$	$b_{13} - e_2$	$b_{14} + e_5$	$b_{15} - e_4$	$b_{16} + e_7$	$b_{17} - e_6$
	e_2	$-b_{12} - e_3$	$-a_2$	$b_{23} + e_1$	$b_{24} + e_6$	$b_{25} - e_7$	$b_{26} - e_4$	$b_{27} + e_5$
	e_3	$-b_{13} + e_2$	$-b_{23} - e_1$	$-a_3$	$b_{34} - e_7$	$b_{35} - e_6$	$b_{36} + e_5$	$b_{37} + e_4$
(2)	e_4	$-b_{14} - e_5$	$-b_{24} - e_6$	$-b_{34} + e_7$	$-a_4$	$b_{45} + e_1$	$b_{46} + e_2$	$b_{47} - e_3$
	e_5	$-b_{15} + e_4$	$-b_{25} + e_7$	$-b_{35} + e_6$	$-b_{45} - e_1$	$-a_5$	$b_{56} - e_3$	$b_{57} - e_2$
	e_6	$-b_{16} - e_7$	$-b_{26} + e_4$	$-b_{36} - e_5$	$-b_{46} - e_2$	$-b_{56} + e_3$	$-a_6$	$b_{67} + e_1$
	e_7	$-b_{17} + e_6$	$-b_{27} - e_5$	$-b_{37} - e_4$	$-b_{47} + e_3$	$-b_{57} + e_2$	$-b_{67} - e_1$	$-a_7$

Let L denote left translation by the vector $u = a + be_1 + ce_2 + de_3 + ee_4 + fe_5 + ge_6 + he_7$.

Then

$\det(L) = (a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2)^3 [a^2 + a_1b^2 + a_2c^2 + a_3d^2 + a_4e^2 + a_5f^2 + a_6g^2 + a_7h^2]$
independent of the b_{ij} 's. We have

Theorem 3. The modified octonion algebra in Table (2) is a division algebra if and only if $a_i > 0$ for $i = 1, \dots, 7$.

Theorem 4. The algebras represented by Table (2) are alternative if and only if all the a_i 's are 1 and all the b_{ij} 's are 0.

Proof. As before,

$$e_i(e_j e_j) = -a_j e_i \text{ and for some } k, (e_i e_j) e_j = (b_{ij} \pm e_k) e_j = b_{ij} \pm b_{ij} e_j - e_i.$$

Since the octonions are alternative, it follows that the algebras given by Table (2) are distinct from the octonions unless all the a_i 's are 1 and all the b_{ij} 's are 0.

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